Evaluation of Finite Element Formulation for One-Dimensional Consolidation

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Abstract—Consolidation process is defined as the progress of which excess pore water pressure generated by external loading dissipated out of the soil pores and subsequently causing the soil to compress. The ability to predict the dissipation of the excess pore water pressure is important in the assessing the performance of foundations. In this paper, a formulation of Finite Element (FE) method is developed for solving uncoupled consolidation problem and its validity is examined. The basic matrix equations for uncoupling analysis using finite elements are derived based on the Galerkin's weighted residual method. Spatial variables are discretized using the same shape function and a fully implicit scheme is used to discretize the time domain. Using the MATLAB program for writing the FE code, the results are compared with that obtained by classic Terzaghi's one dimensional consolidation theory. The FE analysis showed good agreement with the closed form solution.

Index Terms—Consolidation, uncoupled approach,saturated soil, Terzaghi's theory, Finite Element Method (FEM), implicit.

1 Introduction

Soil made up essentially of soil particle with voids in between. The void generally occupied partly by water and partly by air. Saturated soil is a two phase material consisting of a solid skeleton and water filled voids [1]. When an increment of stress is suddenly applied to an element of saturated soil, there is an instantaneous increase in pore pressure and excess pore pressure develops.

The process of consolidation under one-dimensional was first investigated by Terzaghias in[2] and was extended to three dimensional by Biotas in [3]. Both authors agreed that the flow of pore water was governed by Darcy's law and the response of soil skeleton was elastic.

Finite element formulation based on one-dimensional idealization has been established and the numerical characteristics have been examined. Desai [4] compared two alternatives for solution of one-dimensional consolidation which used a finer mesh with lower order approximation function and used a coarser mesh with higher order approximating function. They found that, in terms of accuracy, the linear model shows better agreement with the closed form solution compared to the cubic model. An examination of computational time, the cubic model was found to be a about four times longer than for linear model due to the greater degree of connectivity thus increasing the band width of the equation set solution.

The application of the finite element method to the solution of consolidation equation was considered by Sandhu and Wilson as in [5]. Since then, a number offinite element formulationsfor the consolidation of elastic material have appeared. These include the works ofCarter as in [6], and Menendez as in [7]. Solution techniques for finite element analysis of consolidation are usually based on first order, implicit integration methods. The backward Euler scheme is widely used in both linear and nonlinear studies. The stability of consolidation analyses has received considerable attention by investigators and many interpolation schemes have been proposed for the time domain.

The stability and accuracy of first order integration schemes has been investigated by Booker and Small as in [1]. They proved that the integration parameter, θ must be less than 0.5 in order for the solution scheme to be unconditionally stable. A fully explicit method is obtained when using θ equal to zero. Vermeer and Verrujit [8] investigated the time steps should not be made too small to avoid oscillation in the pore water pressure and they concluded that a lower limit on the time step, based on the coefficient of consolidation and the mesh size, could reduce the occurrence of oscillatory results. In addition, Abid and Pyrah [9] found that the explicit scheme is not recommended and implicit scheme found to be satisfactory and is recommended.

Huang and Griffiths [10] re-examined coupled, uncoupled and the Terzaghi one-dimensional consolidation theories using the finite element method. They concluded, for a layered soil system, combining the coefficient of permeability and coefficient of compressibility into a single coefficient of consolidation will give wrong excess pore water pressure distribution. Recently, Osman [11] found out the uncoupled consolidation analysis in excellent agreement with the coupled analysis and the uncoupled analysis lead to a relatively simpler calculation procedure compared to coupled analysis. Other researcher also concluded that for the case where the flow occurs in one direction when hydraulic conductivity is constant with time, the variation of pore pressure with time calculated from uncoupled analysis becomes identical to that calculated using Terzaghi's uncoupled analysis [12].

In this paper, a formulation of one dimensional consolidation is considered. The finite element is employed to achieve numerical solution, in conjunction with a finite difference

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time-stepping algorithm to define the transient nature of the problem.

2 Theory of Consolidation

The theory, which is expressed in mathematical form, becomes the basis for all the studies of consolidation as well as other subjects related to the deformation of the flow of fluids through the porous media. The following assumptions are essential for the general development of the consolidation theory as first given by Terzaghi:[13]

1. Soil in the consolidating layer is homogenous.
2. Soil is completely saturated (S=100%).
3. Compressibility of either water or soil grains is negligible.
4. Strains are infinitesimal. An element of dimension dx, dy, and dz has the same response as one with dimensions x, y, and z.
5. Flow is one-dimensional.
6. Compression is one-dimensional.
7. Darcy's law is valid (v=ki).
8. Soil properties are constants.
9. The void ratio, e vs. pressure, p response is linear.

For one-dimensional flow (in the vertical direction or z), the volumetric flow is defined as

$$k\frac{\partial^2 h}{\partial z^2}dxdydz=\frac{dV}{dt} \quad (1)$$

The element volume is *dx dydz* and the pore volume is (*dx dydz*) [e / (1+e)]. All volume changes *V* are the pore volume changes from assumption 3, so the time rate of volume change as

$$\frac{\partial}{\partial t}\left(dxdydz\frac{e}{e+1}\right)=\frac{dV}{dt} \quad (2)$$

Since the (*dx dydz*) / (1+e) is the constant volume of solids, Equation (2) can be rewritten as [(*dx dydz*) / (1+e)] (∂e /∂t). Equating this into Equation (1):

$$k\frac{\partial^2 h}{\partial z^2}=\left(\frac{e}{e+1}\right)\frac{\partial e}{\partial t} \quad (3)$$

Only a pressure head in excess of the hydrostatic head will cause flow (and volume change) and since h = Δu/γ_w Equation (3) can be rewritten as

$$\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}=\left(\frac{1}{1+e}\right)\frac{\partial e}{\partial t} \quad (4)$$

From the slope of the linear part of an arithmetic plot of void ratio, e or strain, ε versus pressure, p, the coefficient of compressibility, a_v and the compressibility ratio, a'_v can be defined as

$$a_v=\frac{\Delta e}{\Delta p}=\frac{de}{dp} \quad (5)$$

$$a'_v=\frac{\Delta\varepsilon}{\Delta p}=\frac{d\varepsilon}{dp} \quad (6)$$

With negative sign ignored. Before any pore pressure dissipates, *dp* = *du*, so *de* = $a_v du$, which can then be substituted into equation (4) to give

$$\left[\frac{k(1+e)}{a_v\gamma_w}\right]\frac{\partial^2 u}{\partial z^2}=\frac{\partial u}{\partial t} \quad (7)$$

The bracketed terms can be defined as the coefficient of consolidation, c_v or

$$c_v=\left[\frac{k(1+e)}{a_v\gamma_w}\right] \quad (8)$$

And the coefficient of volume compressibility, m_v (and introducing the initial in situ void ratio,e_0 for e) as

$$m_v=\left[\frac{a_v}{1+e_0}\right]=a'_v \quad (9)$$

$m_v$has the units of stress-strain modulus (kPa or Mpa). It is often referred to as the constrained modulus and is the modulus of deformation, E_s. Rewrite c_v in a form suitable for finite element analysis.

$$c_v=\left[\frac{k(1+e)}{a_v\gamma_w}\right]=\frac{k}{\gamma_w m_v} \quad (10)$$

Rewrite Equation (7) as

$$\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}=m_v\frac{\partial u}{\partial t} \quad (11)$$

The solution of Equation (11) is not trivial and uses of Taylor series expansion. It is as follows:

$$u=\frac{4u_i}{\pi}\sum_0^{\infty}\frac{1}{2N+1}\left[\sin\left(\frac{(2N+1)\pi z}{H}\right)dz\right]e^{-\frac{(2N+1)^2\pi^2}{H^2}c_v t} \quad (12)$$

This equation is general and applies for any case of initial hydrostatic pressure, u_i in a thickness of soil, z and H refers to drainage path. Since the coefficient of consolidation, c_v is constant and time, t is a multiple of c_vH, a dimensionless time factor, T can be defined as

$$T=\frac{c_v t_i}{H^2} \quad (13)$$

The average degree of consolidation as in the following equation:

$$U=1-\frac{8}{\pi^2}\frac{1}{(2N+1)^2}e^{-\frac{(2N+1)^2\pi^2}{H^2}c_v t} \quad (14)$$

3 Finite Element Formulation

To derive the basic matrix equations for consolidation, the starting point is the differential equation. The governing differential equation for consolidation process can be written as

$$m_v\frac{\partial u}{\partial t}-\frac{k}{\gamma_w}\frac{\partial^2 u}{\partial z^2}=0 \quad (15)$$

The spatial discretization of the problem is achieved by using linear approximation function as shown in Figure 1. The clay layer is divided into 20 uniform elements. The element has two nodes with 2 degree of freedom. Since in this paper, the uncoupled approach is considered, the secondary variable (settlement) will be calculated separately once the primary variable (pore pressure) has been obtained as in [9]. The pore pressure is linear variation along the element and can be expressed as:

$$u = N_1 u_1 + N_2 u_2 \quad (16)$$

where u_1 and u_2 denote the nodal pore pressure associated with the shape function of the element;

$$N_1 = 1 - \frac{z}{H}, \; N_2 = \frac{z}{H} \quad (17)$$

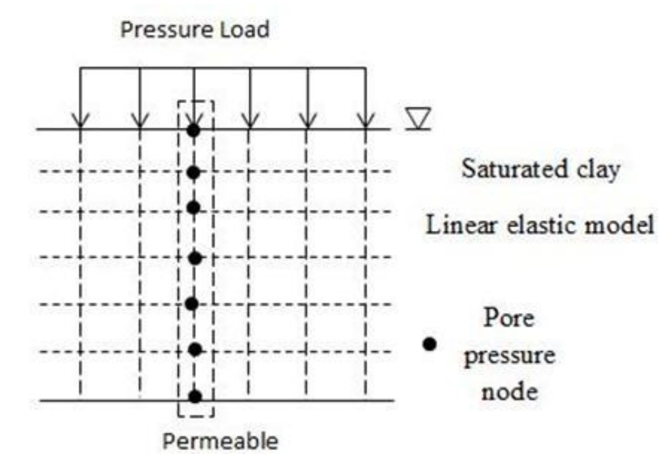

Fig. 1.Finite element discretization of one-dimensional consolidation.

When Equation (16) is substituted into the differential equation of consolidation, the following equation can be obtained.

$$m_v \frac{\partial u}{\partial t} - \frac{k}{\gamma_w}\frac{\partial^2}{\partial z^2}\{N_1 N_2\}\{u_1 u_2\}^T = 0 \quad (18)$$

Following the standard Galerkin's weighted residual approach, a solution of partial differential equation of consolidation can be achieved by multiply the whole Equation (11) with shape function of pore pressure N_1 and N_2, then integrate it over the element. The need to apply integration by part is important in order to decrease the order of the governing differential equations. The residuals are set zero, the following equations can be obtained [14].

$$m_v \int_0^L \{N\}^T \frac{\partial u}{\partial t} dz - \left(\frac{k}{\gamma_w} \{N\}^T \frac{\partial u}{\partial z}\Big|_0^L - \int_0^L \frac{k}{\gamma_w}\frac{dN}{dz}\frac{dN}{dz} u_i dz \right) = 0 \quad (19)$$

Generally, a finite difference solution to the time integration has been used. Booker and Small [1] suggested that θ ≥ ½ can keep the stability of integration scheme. The implicit time integration (Euler backward) is used and θ = 1 is adopted here. Term derivatives of time may be written in finite difference form as:

$$\frac{\partial u}{\partial t} = \frac{u_i - u_{i-1}}{\Delta t}$$

Equation (19) becomes:

$$m_v \int_0^L \{N\}^T \left(\frac{Nu_i - N_{i-1}}{\Delta t} \right) dz + \int_0^L \frac{k}{\gamma_w}\frac{dN}{dz}\frac{dN}{dz} u_i dz = 0 \quad (20)$$

$$\frac{1}{\Delta t} m_v [M]\{u\}_i - \frac{1}{\Delta t} m_v [M]\{u\}_{i-1} + \frac{k}{\gamma_w}[K]\{u\}_i = 0 \quad (21)$$

Rearrange the eqution:

$$\left(\frac{1}{\Delta t} m_v [M] + \frac{k}{\gamma_w}[K] \right)\{u\}_i = \frac{1}{\Delta t} m_v [M]\{u\}_{i-1} \quad (22)$$

where: L is the length of the element,

$$\text{Stiffness matrix, } K = \int_0^L \begin{bmatrix} \frac{dN_1}{dz}\frac{dN_1}{dz} & \frac{dN_1}{dz}\frac{dN_2}{dz} \\ \frac{dN_2}{dz}\frac{dN_1}{dz} & \frac{dN_2}{dz}\frac{dN_2}{dz} \end{bmatrix} dz,$$

$$\text{Mass matrix, } M = \int_0^L \begin{bmatrix} N_1 N_1 & N_1 N_2 \\ N_2 N_1 & N_2 N_2 \end{bmatrix} dz$$

Matrix equations of elements have been obtained for the case of one-dimensional consolidation. Based on that, the program is coded using MATLAB in this paper.

4 NUMERICAL EXAMPLE

The case is a clay layer of thickness 7m subjected to a vertical pressure of 100kPa and maintained constant with time. Drainage is allowed from the top and bottom surface. The clay layer is divided into 20 uniform elements. And the time step of Δt = 0.05 day was used. The following material properties are assumed:

TABLE 1
MATERIAL PROPERTIES

Soil properties	
Modulus elasticity, E	2 MPa
Coefficient of permeability, k	8.53E-4 m/day
Coefficient of compressibility, m_v	4.57E-4 kPa^{-1}
Depth of soil, z	7m

Figure 2(a), 2(b), 2(c), and 2(d), shown a plot of the excess pore pressure with depth from the numerical solution and from Terzaghi's solution, at four different times after applied of loading, i.e. 1 day, 100 days, 365 days and 600 days. The results show very good agreement between the numerical analysis and the exact solution.

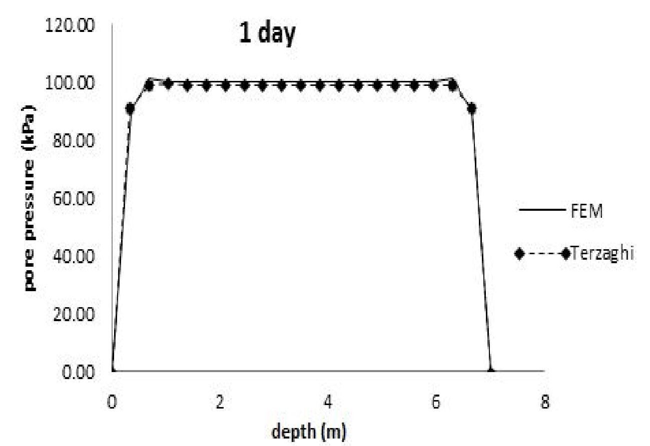

Fig. 2(a).Comparisons of excess pore pressure along depth at time, t =1 day.

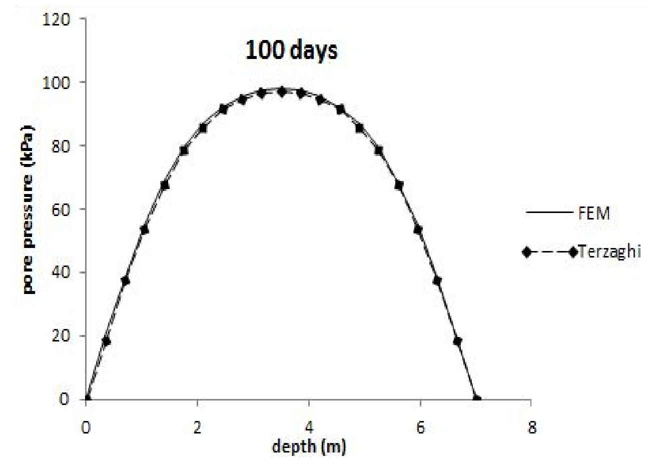

Fig. 2(b).Comparisons of excess pore pressure along depth at time, t =100 days.

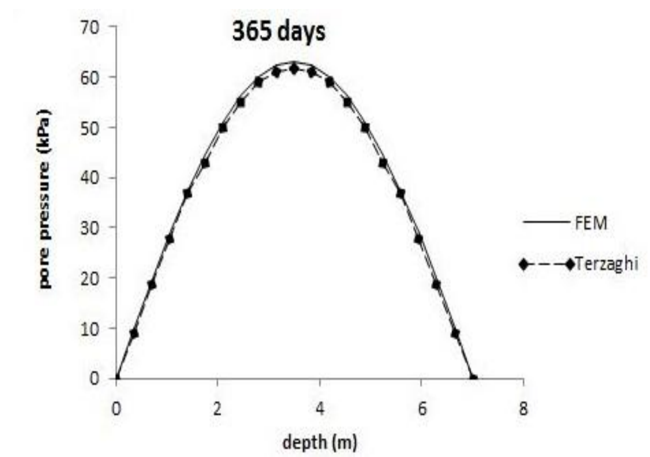

Fig. 2(c).Comparisons of excess pore pressure along depth at time, t =365 days.

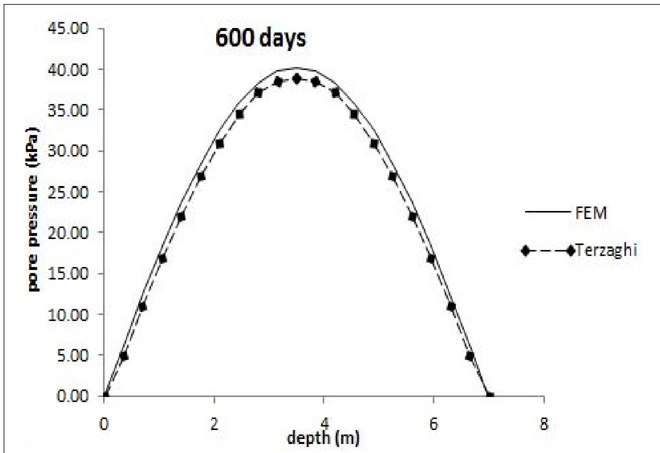

Fig. 2(d).Comparisons of excess pore pressure along depth at time, t =600 days.

Figure 3 is graph of the average degree of consolidation, U versus the dimensionless time factor, Tv. Similar results of degree of consolidation were obtained from the numerical solution and the exact solution of Terzaghi's theory.

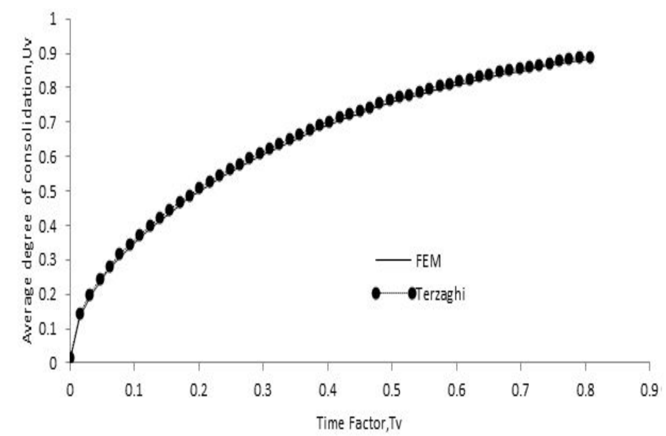

Fig. 3.Average degree of consolidation, U versus time factor, Tv: comparison between Terzaghi's exact solution and FEM.

5 Conclusion

In this study, the problem of the consolidation of an elastic saturated soil was solved with success using finite element method. A formulation consistent with the Terzaghi's one-dimensional theory has been developed and it has been restricted to the case of soils with linear elastic.

The numerical algorithms implemented in this paper are efficient due to the result which shows a good agreement with the Terzaghi's exact solution. Further work is currently undergoing in order to extend this theory to the analysis of soils with non-elastic behaviour and coupled consolidation approach.

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